

Eguchi-Hanson like space-times in $F(R)$ gravity

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Abstract

We consider a model of $F(R)$ gravity in which exponential and power corrections to Einstein- Λ gravity are included. We show that this model has 4-dimensional Eguchi-Hanson type instanton solutions in Euclidean space. We then seek solutions to the five dimensional equations for which space-time contains a hypersurface corresponding to the Eguchi-Hanson space. We obtain analytic solutions of the $F(R)$ gravitational field equations, and by assuming certain relationships between the model parameters and integration constants, find several classes of exact solutions. Finally, we investigate the asymptotic behavior of the solutions and compute the second derivative of the $F(R)$ function with respect to the Ricci scalar to confirm Dolgov-Kawasaki stability.

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I. INTRODUCTION

In recent years, much work has been done in order to bring us more information on extra dimensions. In this domain, investigation of black objects, soliton solutions and instantons are of particular importance, and could play a key role in opening a window to extra dimensions.

In four dimensions, it is well known that some of the static black hole solutions described by the Majumdar-Papapetrou solution [1] can be extended to multi-black objects in higher dimensions [2]. The four dimensional Riemannian manifolds for gravitational instantons can be asymptotically flat, asymptotically locally Euclidean, asymptotically locally flat or compact without boundary. For example, Hawking's interpretation of the Taub-NUT solution [3] is an example of asymptotically locally flat space. The simplest nontrivial example of asymptotically locally Euclidean spaces is the metric of Eguchi-Hanson [4]. Asymptotically locally Euclidean instantons have been found explicitly by Gibbons and Hawking [5] and they are known implicitly through the work of Hitchin [6]. The complex projective space \mathbb{CP}^2 is an example of compact, anti self-dual instanton solving Einstein's equation with a cosmological constant term [7] .

Some what more recently, new soliton solutions with interesting properties were discovered a few years ago in 5-dimensional Einstein gravity [8, 9]. These solutions resemble the Eguchi-Hanson metrics [4] with AdS/Z_p asymptotics. Eguchi-Hanson metrics were originally obtained as solutions to the 4-dimensional Euclidean Einstein equations [11] and have the form

$$ds^2 = \frac{dr^2}{1 - \frac{\alpha^4}{r^4}} + \frac{r^2}{4} ((d\psi + \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta d\varphi^2) \quad (1)$$

where α is a constant of integration. The Riemann curvature tensor of these solutions is self-dual, obeying the relation $\epsilon_{\alpha\beta}{}^{\mu\nu} R_{\mu\nu\rho\tau} = R_{\alpha\beta\rho\tau}$.

Once we entertain the notion of extra dimensions, we need not impose the asymptotic flatness condition, spherical topology of the horizon and other four dimensional restrictions. In fact, higher dimensional solutions admit a variety of asymptotic structures and horizon topologies that have more interesting properties and richer structure than four-dimensional ones. For example, there have been many interesting results in Kaluza-Klein theory [10] with compact extra dimensions as well as in solutions making use of Eguchi-Hanson and Taub-NUT spaces [11, 12] with nontrivial topology and structure.

From a geometrical point of view, one of the simplest modifications of the gravitational interaction to higher order is $F(R)$ gravity, whose action is an arbitrary function of the curvature scalar R [13–16]. A key motivation for considering this class of theories has to do with addressing a number of cosmological problems. These include the power law early-time inflation [17, 18], late-time cosmic accelerated expansion [18–21] and the singularity problem arising in the strong gravity regime [22–26]. Furthermore one can find an explanation of the hierarchy problem [27], the four cosmological phases [28] and the rotation curves of spiral galaxies [29] within $F(R)$ gravity.

Following the method of [30] (which is concerned with a known class of the $F(R)$ theory instead of Einstein gravity) the main scope of this work is to present some results on Eguchi-Hanson like metrics and investigate their interesting properties. Although many of the solutions we obtain have unphysical properties, we find generalizations of the Eguchi-Hanson instanton in 4 dimensions, and a new soliton solution in 5 dimensions.

In contrast with general relativity, $F(R)$ gravity may be considered intrinsically unstable. This difference is due to the fact that the field equations in general relativity are of second order and therefore their trace gives an algebraic equation for Ricci scalar, but the field equations of $F(R)$ gravity are of fourth order and so their trace gives a dynamical equation for Ricci scalar. **It has been noted that if $F(R) \neq 0$ and $dF/dR = 0$ then no stable ground state existed [31]. For the solutions we obtain we find that $F(R) = dF/dR = 0$. Furthermore,** it has been shown that the effective mass of the dynamical field of Ricci scalar is related to the second derivative of the $F(R)$ with respect to R [32]. Therefore, in order for the dynamical field to be stable its effective mass must be positive, a requirement usually referred to as Dolgov-Kawasaki stability criterion. The Dolgov-Kawasaki instability criterion, which has found in the metric version of $F(R)$ theories, is sufficiently strong to veto some models [33].

The outline of our paper is as follows. In section II we present a short review of field equation of d -dimensional $F(R)$ gravity. This field equation is then solved for 4-dimensional Eguchi-Hanson space in Sec. III. We then generalize it to 5-dimensional Eguchi-Hanson like space-time and obtain a new soliton solution. Conclusions are drawn in the last section, and we present more solutions using the same approach in the appendix.

II. BASIC FIELD EQUATIONS

We start from the following action of pure general $F(R)$ gravity

$$\mathcal{I}_G = -\frac{1}{16\pi} \int d^d x \sqrt{-g} F(R), \quad (2)$$

Variation of Eq. (2) with respect to metric $g_{\mu\nu}$, leads to the field equation of $F(R)$ gravity

$$R_{\mu\nu} F_R - \nabla_\mu \nabla_\nu F_R + \left(\square F_R - \frac{1}{2} F(R) \right) g_{\mu\nu} = 0, \quad (3)$$

where $R_{\mu\nu}$ is the Ricci tensor and $F_R \equiv dF(R)/dR$. We consider a recently proposed [30] $F(R)$ model

$$F(R) = R - 2\Lambda - \lambda \exp(-\xi R) + \kappa R^n \quad (4)$$

in which exponential and power corrections to Einstein- Λ gravity are included. This choice of $F(R)$ gravity has some interesting properties such as providing charged solutions from pure $F(R)$ gravity [30].

Viable modifications to Einstein gravity must pass all sorts of empirical tests, from the large scale structure of the universe to galaxy and cluster dynamics to solar system tests. One of the outstanding questions in $F(R)$ gravity is whether it is consistent with solar system tests or not. When the correction term to Einstein gravity is of the exponential form [14, 34], it can be shown that there is no conflict with solar system tests and that the high curvature condition is satisfied [35]. Furthermore, the solutions of this model are virtually indistinguishable from those in general relativity, up to a change in Newton's constant [35].

In addition, one can choose R^2 with exponential corrections to Einstein gravity [34, 36] to explain inflation. By adjusting the free parameters, this model can satisfy the high curvature condition, early universe inflation, stability and the late time acceleration [35]. Also, in order to resolve the singularity problem arising in the strong gravity regime, Kobayashi and Maeda [26] have considered a higher curvature correction proportional to R^n where $n > 1$.

Motivated by the above, we seek instanton solutions to $F(R)$ gravity using (4) for 4-dimensional Eguchi-Hanson space and soliton solutions for Eguchi-Hanson like $(4 + 1)$ - dimensional space-time.

III. GENERALIZED EGUCHI-HANSON INSTANTONS

We consider first the 4-dimensional Eguchi-Hanson type metric ansatz

$$ds^2 = \frac{dr^2}{g(r)} + r^2 (\sigma_x^2 + \sigma_y^2) + r^2 g(r) \sigma_z^2, \quad (5)$$

where the differential one forms σ_i can be expressed in terms of Euler angles θ , ϕ and ψ

$$\begin{aligned} \sigma_x &= \frac{1}{2} (\sin \psi d\theta - \sin \theta \cos \psi d\phi), \\ \sigma_y &= \frac{1}{2} (-\cos \psi d\theta - \sin \theta \sin \psi d\phi), \\ \sigma_z &= \frac{1}{2} (d\psi + \cos \theta d\phi). \end{aligned} \quad (6)$$

For $F(R) = R$, it is well known that the solution to the Euclidean Einstein equations is [4]

$$g(r) = 1 - \frac{\alpha^4}{r^4} \quad (7)$$

where α is constant. For $\alpha = 0$ the constant- r surfaces of the metric (5) have the geometry S^3 , whereas for $\alpha \neq 0$ the geometry must be S^3/\mathbb{Z}_2 to avoid singularities. Note that the metric $\sigma_x^2 + \sigma_y^2$ yields the geometry of a 2-sphere of radius $1/2$.

We wish to obtain general solutions using the ansatz (5) for the mentioned model of $F(R)$ theory, and investigate their geometrical properties.

Since the metric is purely Euclidean with no time-like direction, any solutions we obtain cannot be interpreted as black holes. If the function $g(r)$ vanishes at $r = r_0$ then the metric will have a singularity unless ψ is appropriately periodically identified with period

$$\Delta\psi = \frac{16\pi}{r_0 g'(r_0)} \quad (8)$$

where $g' \equiv dg/dr$.

Using Eqs. (5) and (3), we find the solution

$$g(r) = 1 - \frac{\chi r^2}{24} + \frac{q^2}{r^2} + \frac{a}{r^4}, \quad (9)$$

$$\begin{aligned} \xi &= \frac{\Pi}{\Omega}, \\ \lambda &= -\Omega e^{\frac{\chi\Pi}{\Omega}}, \end{aligned} \quad (10)$$

for the metric function, where $\Pi = n\kappa\chi^{n-1} + 1$, $\Omega = 2\Lambda - \kappa\chi^n - \chi$, and χ , q and a are integration constants and ξ , β , κ and n are free parameters. In order to study the geometrical

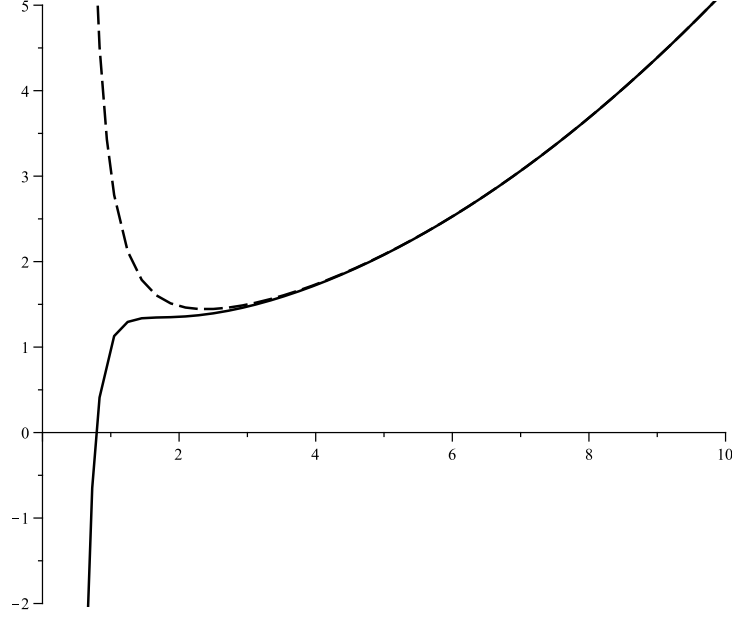


FIG. 1: Eq. (9): $g(r)$ versus r for $\chi = -1$, $q = 1$, and $a = -1$ (solid line) and $a = 1$ (dashed line).

structure of this solution, we first look for the essential singularity(ies). The Ricci scalar and the Kretschmann scalar are

$$R = \chi, \quad (11)$$

and

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{384a^2}{r^{12}} + \frac{384aq^2}{r^{10}} + \frac{128q^4}{r^8} + \frac{\chi^2}{3}, \quad (12)$$

which means that the Kretschmann scalar (12) diverges at $r = 0$, is finite for $r \neq 0$ and approaches $\frac{\chi^2}{3}$ as $r \rightarrow \infty$.

For $\chi < 0$ and positive a this solution has a singularity at $r = 0$. However for $\chi < 0$ and negative a the function $g(r)$ vanishes at a finite value of r we denote by r_0 . The space is geodesically complete provided the coordinate ψ has period

$$\frac{64\pi r_0^2}{3|\chi|r_0^4 + 8q^2 + 16r_0^2}$$

where

$$\frac{3|\chi|r_0^6}{24} + r_0^4 + q^2r_0^2 - |a| = 0$$

Figure 1 illustrates that for negative a , the function $g(r)$ vanishes at a certain value of r ; this solution is like a “bubble” in Euclidean space or a kind of soliton.

For $\chi > 0$ and positive a the function $g(r)$ has a real root at r_0 , becoming negative for larger values of r . This space has a naked singularity at $r = 0$. However if a is also negative

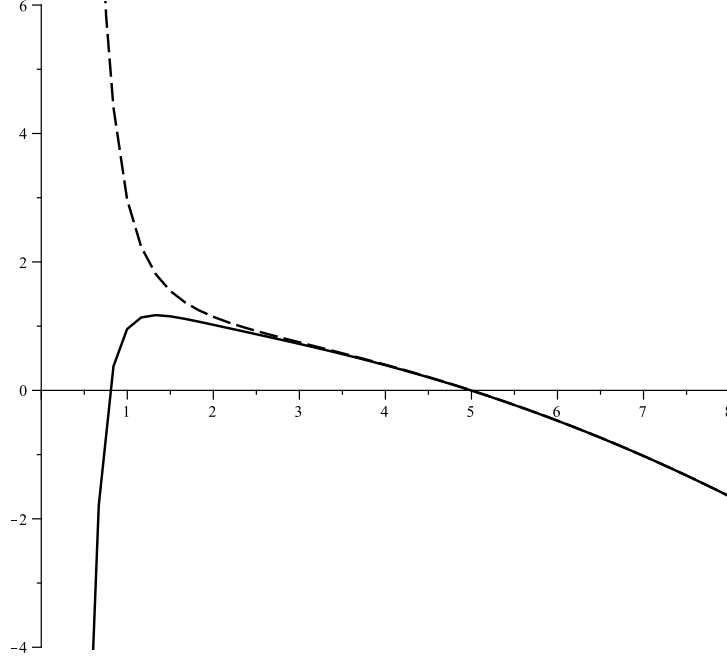


FIG. 2: Eq. (9): $g(r)$ versus r for $\chi = 1$, $q = 1$, and $a = -1$ (solid line) and $a = 1$ (dashed line).

then the metric function $g(r)$ has two roots, $r_<$ and $r_>$, which delineate the range of r (see Fig. 2). The solution will be free of singularities provided the period of ψ at $r_<$ is the same as that at $r_>$. We find that there are no real values of the parameters for which this situation holds, and so conclude that all $\chi > 0$ solutions are singular.

This solution is problematic insofar as the circle described by ψ is growing exponentially relative to the 2-sphere as $r \rightarrow \infty$. The reason is that $g(r)$ is growing like r^2 for large r , so the ψ -circle is growing much faster than the 2-sphere for large r . This will not happen if $\chi = 0$, which implies $\xi = \frac{1}{2\Lambda}$ and $\lambda = -2\Lambda$.

We therefore conclude that $\chi = 0$ and $a < 0$ yields the only physically reasonable class of instanton solutions, whose metric is

$$ds^2 = \frac{dr^2}{1 + \frac{q^2}{r^2} - \frac{\alpha^4}{r^4}} + r^2 (\sigma_x^2 + \sigma_y^2) + r^2 \left(1 + \frac{q^2}{r^2} - \frac{\alpha^4}{r^4} \right) \sigma_z^2, \quad (13)$$

where we have set $a = -\alpha^4$. We find that the metric function vanishes at $r_0 = \frac{\sqrt{2}}{2}(\sqrt{q^4 + 4\alpha^4} - q^2)$ and that

$$\Delta\psi = 4\pi \left(1 - \frac{q^2}{\sqrt{q^4 + 4\alpha^4}} \right) \quad (14)$$

generalizing the Eguchi-Hanson metric (7) to $F(R)$ gravity. Note that the class of metrics

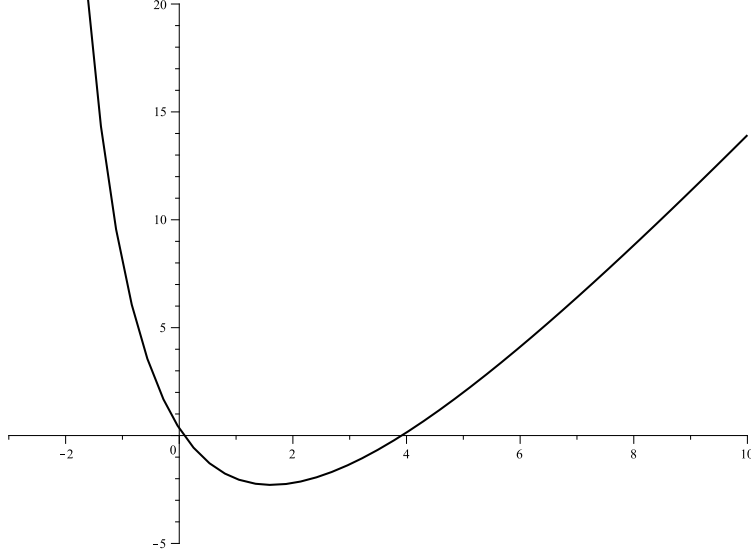


FIG. 3: Eq. (15): F_{RR} versus κ for $\chi = -1$, $\Lambda = 1$ and $n = 3$.

(13) cannot be obtained by Wick rotation of some coordinate in a corresponding Lorentzian-signature space-time.

Although we have not included a time-like coordinate yet, the $F(R)$ model can be checked for stability. In order to check the stability condition (Dolgov-Kawasaki stability), we obtain the second derivative of the $F(R)$ function with respect to Ricci scalar

$$F_{RR} = \frac{\Pi^2}{\Omega} + n(n-1)\kappa\chi^{n-2}, \quad (15)$$

which shows that this model is stable provided the free parameters of the model are chosen suitably (see Fig. 3 for more details). We should note that for $\chi = 0$, F_{RR} reduces to

$$F_{RR} = \frac{1}{2\Lambda}. \quad (16)$$

which confirms that this solution is stable for $\Lambda > 0$.

IV. 5-DIMENSIONAL SOLITON SOLUTIONS

Here, we include the time coordinate and begin with the following 5-dimensional ansatz for the metric

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{h(r)g(r)} + r^2(\sigma_x^2 + \sigma_y^2) + r^2g(r)\sigma_z^2. \quad (17)$$

In this metric, the hypersurface $t = \text{constant}$ corresponds to Eguchi-Hanson space when we fix $h(r) = 1$ and set $g(r)$ as in (7). For $g(r) = 1$, Eq. (17) reduces to

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{h(r)} + \frac{r^2}{4} (d\theta^2 + d\phi^2 + d\psi^2) + \frac{r^2}{2} \cos(\theta) d\phi d\psi, \quad (18)$$

which is Schwarzschild-like space-time. When we choose $h(r) = 1$, one can find that $g(r)$ is the same as Eq. (9) with the same analysis. In this section, we are looking for exact solutions of the field equations (3) for the Eguchi-Hanson like metric ansatz (17) with $F(R)$ given by Eq. (4).

We should note that for any possible solution for the metric ansatz (17), one may encounter several cases of asymptotic structures and singularities of the metric:

- $h(r)$ flips sign at one or more values of r and $g(r)$ does not change sign for sufficiently large r . These correspond to the usual kinds of horizons: black hole and cosmological. If $h(r) \rightarrow 1$ at large r then there will be no cosmological horizon. There will be a black hole if $h(r)$ flips sign at small r before $g(r)$ does.
- $g(r)$ flips sign at one or more values of r and $h(r)$ does not change sign. At small finite $r > 0$ this will be the edge of the soliton (or bubble), and if $g(r) \rightarrow 1$ at large r then the space-time will be a soliton. If $g(r)$ flips sign again for large r then the space will be compact.
- $g(r)$ flips sign at one or more values of r and $h(r)$ changes sign at other values of r . If $g(r_+) = 0$ and $h(r_c) = 0$ then this is a soliton surrounded by a cosmological horizon. If $h(r)$ has multiple roots, all larger than r_+ , then the soliton is in some kind of black hole that has multiple horizons; any roots in the range $r < r_+$ don't matter because the soliton covers them up.
- $g(r)$ and $h(r)$ both flip sign at the same value of r . In this case the coordinate ψ becomes time-like and the space-time has closed time-like curves (CTCs) beyond this value of r (either larger or smaller, depending on where $g(r)$ and $h(r)$ are both positive).

The only way to get a sensible AdS, or dS, flat structure asymptotically is if $g(r) \rightarrow 1$ and $h(r) \rightarrow \pm r^2$ or $h(r) \rightarrow 1$ for large r – anything else gives some other geometry. For example if $h(r) \rightarrow \pm r^p$ (and $g(r) \rightarrow 1$), then the geometry will be asymptotic to a Lifshitz geometry.

1. *First solution set:*

We find a more interesting class of space-times if we set $\lambda = \frac{e^{\xi\chi}[3\chi-10\Lambda-\kappa(2n-5)\chi^n]}{2\xi\chi+5}$. Solving the field equations (3) and (17) we find

$$\begin{aligned} g(r) &= 1 - \frac{a^4}{r^4}, \\ h(r) &= 1 - \frac{\chi}{20}r^2. \end{aligned} \tag{19}$$

After some algebraic manipulation, we find that the Kretschmann and the Ricci scalars are

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{384a^8}{r^{12}} - \frac{96\chi a^8}{5r^{10}} + \frac{9a^8\chi^2}{50r^8} + \frac{\chi^2}{10}, \tag{20}$$

$$R = \chi, \tag{21}$$

indicating a curvature singularity at $r = 0$, and a space-time of constant curvature as $r \rightarrow \infty$. The singularity in (20) is not accessible from the space-time, which is free of singularities provided the period of ψ is appropriately chosen [8]. This solution is the same as the Eguchi-Hanson soliton [8, 9], and will therefore share all of its properties.

To investigate Dolgov-Kawasaki stability, we compute derivative of the $F(R)$ function

$$F_{RR} = \frac{[\kappa(2n-5)\chi^n - 3\chi + 10\Lambda]\xi^2}{2\xi\chi + 5} + n(n-1)\kappa\chi^{n-2}, \tag{22}$$

and see that for suitable values of free parameters this model is stable (see Fig. 4).

2. *Second solution set:*

Here, we produce another set of solutions with some limitation on the free parameters of the model. We will see that for a certain choice of parameters we obtain a new soliton solution that generalizes the Eguchi-Hanson soliton.

Choosing the following parameters

$$\begin{aligned} n &= 0, \\ \lambda &= e^{\frac{-\chi}{\kappa+\chi-2\Lambda}} (\kappa + \chi - 2\Lambda), \\ \xi &= \frac{-1}{(\kappa + \chi - 2\Lambda)}, \end{aligned} \tag{23}$$

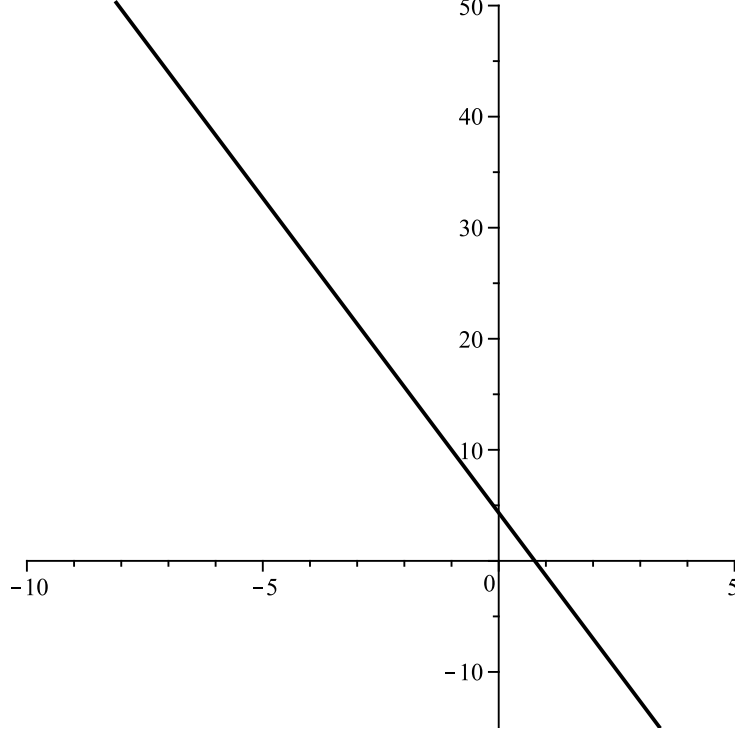


FIG. 4: Eq. (22): F_{RR} versus κ for $\chi = -1$, $\Lambda = 1$, $n = 3$ and $\xi = 1$.

we find a new solution set of the field equation (3) with metric function (17)

$$\begin{aligned} g(r) &= 1 - \frac{5M\chi}{96} - \frac{\chi r^2}{24} - \frac{b}{r^3\sqrt{r^2-M}}, \\ h(r) &= 1 - \frac{M}{r^2}, \end{aligned} \quad (24)$$

where M , χ and b are integration constants, and we have redefined $\kappa - 2\Lambda$ as an effective cosmological constant Λ_{eff} .

Unless $\chi = 0$ the metric will not have the desired asymptotic behavior, and so we make this choice, yielding an asymptotically (locally) flat metric. Note that for positive M this solution is real for all $r > M$; at $r = M$ it is singular. However this singularity is excised from the space-time for all $b > 0$, since the function $g(r)$ has a single root at $r > M$, and this root is larger than the positive root of $h(r)$, as illustrated in figure 5.

The metric is

$$ds^2 = - \left(1 - \frac{M}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{M}{r^2}\right) \left(1 - \frac{b}{r^3\sqrt{r^2-M}}\right)} + r^2 (\sigma_x^2 + \sigma_y^2) + r^2 \left(1 - \frac{b}{r^3\sqrt{r^2-M}}\right) \sigma_z^2 \quad (25)$$

and is an interesting new generalization of the EH soliton. The solution is nonsingular and

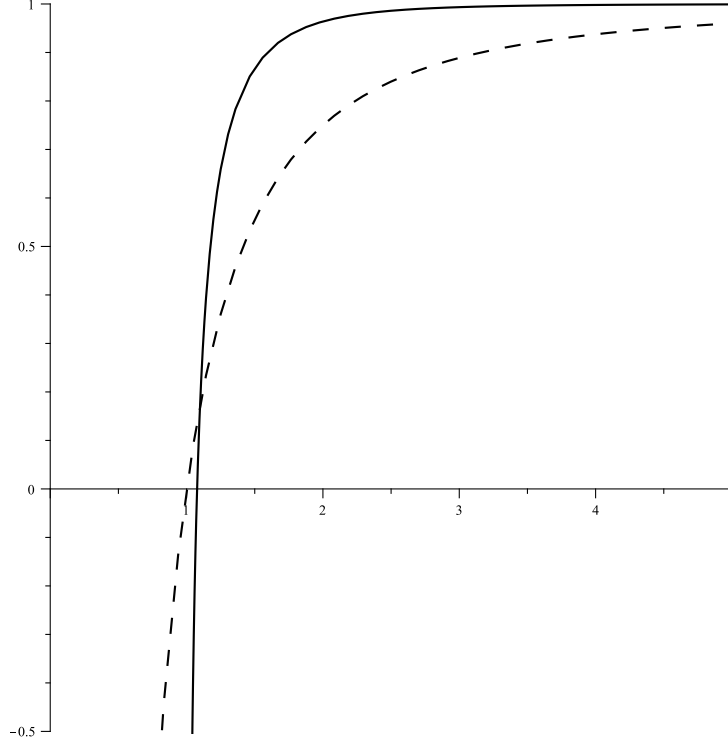


FIG. 5: Eq. (24): $g(r)$ (solid line) and $h(r)$ (dashed line) versus r for $\chi = 0$, $M = 1$ and $b = 0.5$.

the quantity M can be of either sign (see figure 6 for the behavior of the metric functions for $M < 0$), with the size of the soliton an increasing function of M .

The soliton radius r_s is given by solving a quartic equation, which yields

$$r_{s\pm} = \sqrt{|M|} \left[\text{sgn}(M) \left(\frac{1}{4} + \frac{1}{12} \sqrt{9 + 6Y^{1/3} - 288\sigma^8 Y^{-1/3}} \right) + \frac{\sqrt{6}}{12} \left(\sqrt{3 - Y^{1/3} + 48\sigma^8 Y^{-1/3} + \frac{9}{\sqrt{9 + 6Y^{1/3} - 288\sigma^8 Y^{-1/3}}}} \right) \right]^{1/2} \quad (26)$$

where $\sigma = b^4/M^2$ and $Y = 12\sigma^8\sqrt{768\sigma^8 + 81}$. Each of $r_{s\pm}$ are increasing functions of σ , growing linearly with σ for σ sufficiently large.

To ensure regularity of the soliton, the periodicity of ψ must be chosen so that

$$\Delta\psi = \frac{8\pi r_s \sqrt{r_s^2 - M}}{(4r_s^2 - 3M)} = \frac{4\pi}{p} \quad (27)$$

where p is an integer, the latter equality following from the elimination of string singularities at the poles. Note that if $M = 0$ then $p = 2$, consistent with the Eguchi-Hanson instanton.

The above relation constrains the value of σ for any integer p . It is straightforward to show that solutions exist for all integer values of $p > 1$. If $M > 0$ then the solution

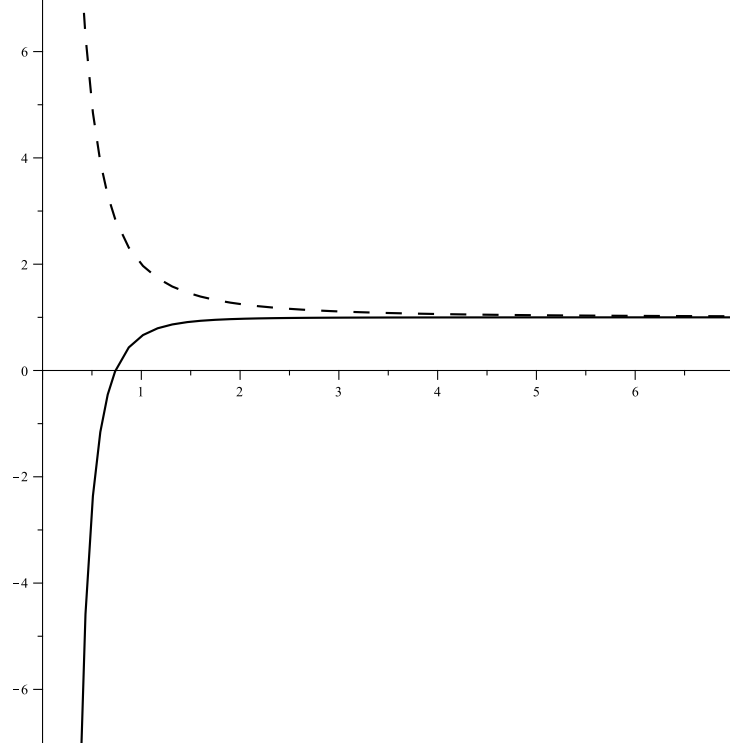


FIG. 6: Eq. (24): $g(r)$ (solid line) and $h(r)$ (dashed line) versus r for $\chi = 0$, $M = -1$ and $b = -0.5$.

$\sigma = 0.720202$ for $p = 2$, decreasing as p increases, whereas if $M < 0$ then the solution $\sigma = 1.090507733$ for $p = 2$, decreasing toward unity as p increases.

The ADM mass of this solution can be calculated using, for example, the background subtraction method, and is easily shown to be $\mathcal{M} = 3\pi M/8p$.

3. Third solution set:

Here we present second solution of Eqs. (3) and (17) with $R = \chi$ as

$$\begin{aligned} g(r) &= \frac{-3\chi}{10C} - \frac{a^4}{r^4} + \frac{\tilde{b}}{r^4\sqrt{Cr^2+6}} + \frac{4(10C+3\chi)}{5C^2r^2}, \\ h(r) &= 1 + \frac{C}{6}r^2, \end{aligned} \tag{28}$$

where we should adjust

$$\begin{aligned} \kappa &= 0, \\ \lambda &= e^{\frac{-\chi}{\chi-2\Lambda}} (\chi - 2\Lambda), \end{aligned}$$

$$\xi = \frac{-1}{(\chi - 2\Lambda)} \quad (29)$$

which confirm that considering Eq. (28), leads to vanishing the R^n correction. For this solution to have physically reasonable asymptotic properties, we must set $\chi = -10C/3$, yielding

$$\begin{aligned} g(r) &= 1 - \frac{a^4}{r^4} + \frac{b}{r^4 \sqrt{r^2/\ell^2 + 1}}, \\ h(r) &= 1 + \frac{r^2}{\ell^2}, \end{aligned} \quad (30)$$

where to have real solutions, we have set $C = 6/\ell^2 > 0$, and we have rescaled $\tilde{b} \rightarrow b$ for convenience. If $b > 0$ then the metric function $g(r)$ has one real root at $r = r_s > 0$, given by the square root of the real solution to the equation

$$(r_s^4 + b)^2 = a^8(r_s^2/\ell^2 + 1)$$

which is a quartic equation in $\sqrt{r_s}$. For large r this space-time is asymptotic to the Eguchi-Hanson soliton [8]. The curvature diverges at $r = 0$ (and hence not at a point located within the space-time). For large values of r , one obtains

$$\lim_{r \rightarrow \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{40}{l^4}. \quad (31)$$

Regularity in the (r, ψ) section and also elimination of string singularities at the north and south poles implies that

$$\Delta\psi = \frac{8\pi r_s^2 \ell \sqrt{r_s^2 + l^2}}{5r_s^4 + 4r_s^2 l^2 - a^4} = \frac{4\pi}{p} \quad (32)$$

Here, we analyze the Dolgov-Kawasaki stability. It is easy to show that

$$F_{RR} = \frac{l^2}{20 + 2\Lambda l^2}, \quad (33)$$

and therefore we conclude that this solution is stable for $\Lambda > -\frac{10}{l^2}$.

It is straightforward to show using the methods of Ref. [37] that the mass of this solution is $\mathcal{M} = -\pi a^4/8\ell^2 p$, the same value as for the Eguchi-Hanson soliton [8], apart from a constant factor due to the Casimir energy.

V. CONCLUSIONS

In this work, we have considered a kind of well-known $F(R)$ gravity in Eguchi-Hanson space and Eguchi-Hanson like space-time. We have shown that, in 4-dimensional Eguchi-Hanson type Euclidean metric, the solutions can be interpreted as stable instantons.

Upon including the time coordinate we found two distinct generalized 5-dimensional Eguchi-Hanson space-times in $F(R)$ gravity, one that is asymptotically flat and another that is asymptotically AdS. These are obtained by imposing additional constraints on the $F(R)$ model parameters. We have investigated their asymptotic behavior, computed their masses, and obtained constraints on the parameters to ensure regularity of the metric. Our investigation of Dolgov-Kawasaki stability of these solutions indicates that their objects are stable.

It is possible to obtain more solution sets for Eguchi-Hanson like (17) space-times using our approach. However an inspection of their basic properties indicates that they are not physical (see appendix for more details).

It would be worthwhile to investigate the thermodynamic as well as dynamical stability of solutions we have found. Other interesting questions, such as whether or not the sub-class of theories yielding the EH-like metrics we have found obey Birkhoff's theorem [38], remain interesting problems for future consideration.

Acknowledgments

This work has been supported financially by Research Institute for Astronomy & Astrophysics of Maragha (RIAAM) under research project No. 1/2348. The work of R. B. Mann has been supported by the Natural Science and Engineering Research Council of Canada.

APPENDIX

A2: Fourth and fifth solution sets:

We produce here two solution sets with the same limitation on the free parameters. Considering the model parameters in the third solution set, one can find two additional solution sets of the field equation (3) with metric function (17) as follows

$$IV : \begin{cases} g(r) = 1 - Cr, \\ h(r) = \frac{2\chi}{41C}r - \frac{2(41C^2 - 12\chi)}{533C^2} - \frac{6(205C^2 - 8\chi)}{2665C^3} \left(\frac{2}{r} + \frac{1}{Cr^2} \right), \end{cases} \quad (34)$$

$$V : \begin{cases} g(r) = \frac{2\chi}{41C}r + \frac{5\chi}{82C^2} - \frac{8(82C^2 - 5\chi)}{451C^3} \left(\frac{1}{r} + \frac{3}{Cr^2} \right), \\ h(r) = 1 - Cr, \end{cases} \quad (35)$$

where χ and C are integration constant. Since $n = 0$, and as we noted for third solution set, one can redefine $\kappa - 2\Lambda$ as an effective cosmological constant.

Calculating the Kretschmann scalar leads to the following asymptotics

$$\begin{aligned} \lim_{r \rightarrow 0} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} &\longrightarrow \infty, \\ \lim_{r \rightarrow \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} &= \frac{219\chi^2}{1681}. \end{aligned} \quad (36)$$

Considering solution III with negative C , one can find that $h(r)$ flips sign at large r and $g(r)$ doesn't, and so we have a cosmological horizon. Respectively adjusting χ to $\frac{2050C^2}{197}$ and $\frac{5248C^2}{419}$ in solutions III and IV, we find that these space-times have a causal horizon since the functions $g(r)$ and $h(r)$ both have the same root in $r = 1/C$.

One can note that these kinds of solutions both have the unphysical asymptotic structure. Here, we compute second derivative of the $F(R)$ function for the mentioned three sets

$$F_{RR} = \frac{-1}{\kappa + \chi - 2\Lambda}, \quad (37)$$

which confirm that this solution is stable for $\Lambda > \frac{1}{2}(\kappa + \chi)$.

A3: Sixth and seventh solution sets:

Looking at Eqs. (3) and (17), and considering Eq. (10), one can obviously obtain two solution sets with constant curvature scalar ($R = \chi$) as follows

$$VI : \begin{cases} g(r) = 1 - \frac{C}{6}r^2, \\ h(r) = \frac{3\chi - C}{11C} + \frac{54(\chi - 4C)}{55C^2r^2}, \end{cases} \quad (38)$$

$$VII : \begin{cases} g(r) = \frac{\chi}{22Cr^2} - \frac{4}{5Cr^4}, \\ h(r) = 1 - Cr^4, \end{cases} \quad (39)$$

where κ and n are free.

For the sixth solution set (Eq. (38)), calculations show that the Kretschmann scalar diverges at $r = 0$. It is more interesting to note that notwithstanding previous solutions, it depends on two parameters as $r \rightarrow \infty$

$$\lim_{r \rightarrow \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{1}{121} \left(27\chi^2 + \frac{56\chi C}{3} + \frac{416C^2}{3} \right). \quad (40)$$

Hence there is a curvature singularity at $r = 0$. This singularity will be hidden behind an event horizon provided either

$$4C > \chi > \frac{C}{3}$$

provided C and χ are both positive. However the space-time ends at the coordinate value $r = \sqrt{6/C}$, and so does not describe a black hole with familiar asymptotic properties.

For C and χ both negative there will be an event horizon cloaking the singularity provided

$$|\chi| > 4|C|$$

In this case the space-time geometrically consists of a squashed 3-sphere whose squashing parameter grows exponentially with r .

The seventh solution set (Eq. (39)) has a curvature singularity at $r = 0$; for large r , we obtain

$$\lim_{r \rightarrow \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{27\chi^2}{121}. \quad (41)$$

Note that the circle described by ψ is growing exponentially relative to the 2-sphere as $r \rightarrow \infty$ and so these solutions both have the same types of problems as the instanton solutions. In order to check the Dolgov-Kawasaki stability, we should obtain F_{RR} . It is easy to show that F_{RR} is the same as Eq. (15).

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